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Lagrangians and Euler morphisms on fibered-fibered frame bundles from projectable-projectable classical linear connections

Abstract. We classify all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators A transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into rth order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $L^{\text{fib-fib}}(Y)$ on Y. Moreover, we classify all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators B transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into Euler morphism $B(\nabla)$ on $L^{\text{fib-fib}}(Y)$. These classifications can be expanded on the kth order fibered-fibered frame bundle $L^{\text{fib-fib},k}(Y)$ instead of $L^{\text{fib-fib}}(Y)$.

1. Introduction. Lagrangians and Euler morphisms are important tools in the variational calculus. Several physical theories are using Euler–Lagrange equations, which are related with the Euler morphism of an rth order Lagrangian on a fibered manifold.

The idea of Lagrangians and Euler morphisms in the case of fibered manifolds was described in [2]. The aim of the present note is the generalization of results which were reached in [1] to the case of fibered-fibered manifolds.

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2. Fibered-fibered manifolds. A fibered-fibered manifold Y is any commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ N \arrow[r, "\pi_0"] & M \end{tikzcd}$$

where maps π, π_0, q, p are surjective submersions and induced map $Y \to X \times_M N$, $y \mapsto (\pi(y), q(y))$ is a surjective submersion. A fibered-fibered manifold has dimension (m_1, m_2, n_1, n_2) if $\dim Y = m_1 + m_2 + n_1 + n_2$, $\dim X = m_1 + m_2$, $\dim N = m_1 + n_1$, $\dim M = m_1$. For two fibered-fibered manifolds Y_1, Y_2 of the same dimension (m_1, m_2, n_1, n_2), a morphism $f\colon Y_1 \to Y_2$ is quadruple of local diffeomorphisms $f\colon Y_1 \to Y_2$, $f_1\colon X_1 \to X_2$, $f_2\colon N_1 \to N_2$, $f_0\colon M_1 \to M_2$ such that all squares of the cube in question are commutative [3].

All fibered-fibered manifolds of the given dimension (m_1, m_2, n_1, n_2) and their all morphisms form the category which we denote by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$.

Every object from the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ is locally isomorphic to the standard fibered-fibered manifold

$$\begin{tikzcd} \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \arrow[r] \arrow[d] & \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \arrow[d] \\ \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \arrow[r] & \mathbb{R}^{m_1} \end{tikzcd}$$

which we denote by $\mathbb{R}^{m_1,m_2,n_1,n_2}$, where arrows are obvious projections.

For fibered-fibered manifold Y we have the fibered-fibered linear frame bundle

$$L^{\text{fib-fib}}(Y) = \left\{ j^1_{(0,0,0,0)}\psi \mid \psi\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to Y \text{ is an } \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\text{-map} \right\}$$

with the jet target projection $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$,

$$\pi_Y(j^1_{(0,0,0,0)}\psi) = \psi(0,0,0,0),$$

where $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$. The bundle $L^{\text{fib-fib}}(Y)$ is a principal bundle over Y with a structure group $G^1_{m_1,m_2,n_1,n_2} = L^{\text{fib-fib}}_{(0,0,0,0)}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ acting on the right on $L^{\text{fib-fib}}(Y)$ by the composition of jets. Every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f\colon Y \to Y_1$ induces a fibered map (a principal bundle morphism) $L^{\text{fib-fib}}(f)\colon L^{\text{fib-fib}}(Y) \to L^{\text{fib-fib}}(Y_1)$ over f by the composition of jets $L^{\text{fib-fib}}(f)\big(j^1_{(0,0,0,0)}\psi\big) = j^1_{(0,0,0,0)}(f \circ \psi)$. The correspondence $L^{\text{fib-fib}}\colon \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2} \rightsquigarrow \mathcal{FM}$ is a bundle functor [2].

3. Lagrangians and natural operators transforming connections into Lagrangians. An rth order Lagrangian on a fibered manifold p :

$X \to M$ is a base preserving morphism $\lambda\colon J^r(X) \to \wedge^m T^*M$ from the r-jet prolongation bundle

$$J^r(X) = \{j^r_x\sigma \mid \sigma\colon M \to X \text{ is a local section of } p\colon X \to M,\ x \in M\}$$

into the bundle $\wedge^m T^*M$ of $m = \dim M$-forms on M [2].

A classical linear connection $\widetilde{\nabla}$ on a fibered manifold $p\colon X \to M$ is projectable if there exists a (unique) classical linear connection $\underline{\widetilde{\nabla}}$ on M such that a connection $\underline{\widetilde{\nabla}}$ is p-related with a connection $\widetilde{\nabla}$, that is $Tp \circ (\widetilde{\nabla}_W Z) = (\underline{\widetilde{\nabla}}_{\underline{W}} \underline{Z}) \circ p$, where W and Z are projectable vector fields on X, which are p-related with vector fields $\underline{W}$ and $\underline{Z}$ on M.

Let Y be a fibered-fibered manifold

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{\pi_0} & M \end{array}$$

We say that a projectable classical linear connection ∇ on Y is projectable-projectable if there exists a unique projectable classical linear connection $\underline{\nabla}$ on X such that a connection $\underline{\nabla}$ is π-related with a connection ∇.

In this paper we study the problem how a projectable-projectable torsion-free classical linear connection ∇ on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) can induce an rth order Lagrangian $A(\nabla)$ on $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ in the canonical way. To this aim we must determine $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators

$$A\colon Q^{\text{proj-proj}}_{\tau} \to \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big),$$

where $m = m_1 + m_2 + n_1 + n_2$, in the sense of [2].

We describe completely all such natural operators A in question.

An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$A\colon Q^{\text{proj-proj}}_{\tau} \to \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big)$$

(where $m = m_1 + m_2 + n_1 + n_2$) sending projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifolds Y of dimension (m_1, m_2, n_1, n_2) into rth order Lagrangians $A(\nabla)$ on the fibered-fibered linear frame bundle $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ for Y is the family of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$A_Y\colon Q^{\text{proj-proj}}_{\tau}(Y) \to Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$$

for $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects Y, where $Q^{\text{proj-proj}}_{\tau}(Y)$ is the space of all projectable-projectable torsion-free classical linear connections on Y and $Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$ is the space of all rth order Lagrangians on

$$\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y.$$

The invariance means that if $\nabla \in Q_\tau^{\text{proj-proj}}(Y)$ and $\nabla_1 \in Q_\tau^{\text{proj-proj}}(Y_1)$ are f-related with respect to an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $f\colon Y \to Y_1$, then $A_Y(\nabla)$ and $A_{Y_1}(\nabla_1)$ are also f-related. The regularity of a natural operator A_Y means that A_Y transforms smoothly parametrized families of connections in $Q_\tau^{\text{proj-proj}}(Y)$ into smoothly parametrized families of Lagrangians in $Lagr^r\big(L^{\text{fib-fib}}(Y)\big)$.

To present an example of natural operator A in question we need some preparations. Let M be an m-manifold with a torsion-free classical linear connection $\underline{\widetilde{\nabla}}$. Given a linear frame $\underline{l} \in L(M)$, the linear isomorphism $\underline{l}\colon \mathbb{R}^m \to T_{\underline{x}}(M)$ defines a coordinate system in $T_{\underline{x}}(M)$. Therefore, $\underline{\widetilde{\nabla}}$-exponential map $Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\colon T_{\underline{x}}(M) \to M$ being the diffeomorphism, defines a normal coordinate system $\underline{\varphi}$ with center $\underline{x}$ in M by the composition $\underline{\varphi} = \underline{l}^{-1} \circ \big(Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\big)^{-1}\colon M \to \mathbb{R}^m$. If $\underline{\varphi}_1\colon M \to \mathbb{R}^m$ is another $\underline{\widetilde{\nabla}}$-normal coordinate system on M with center $\underline{x}$, then $\underline{\varphi}_1 = I \circ \underline{\varphi}$ for some linear isomorphism $I\colon \mathbb{R}^m \to \mathbb{R}^m$ [4].

Let $p\colon X \to M$ be a fibered manifold of dimension (m,n) and let $x \in X_{\underline{x}}$, $\underline{x} \in M$. Let $\widetilde{\nabla}$ be a projectable torsion-free classical linear connection on X with the underlying torsion-free classical linear connection $\underline{\widetilde{\nabla}}$ on M. Since a connection $\widetilde{\nabla}$ is p-related with $\underline{\widetilde{\nabla}}$, then p sends $\widetilde{\nabla}$-geodesics into $\underline{\widetilde{\nabla}}$-geodesics. Consequently, the $\widetilde{\nabla}$-exponential map $Exp_x^{\widetilde{\nabla}}\colon T_x(X) \to X$ at x is a local fibered diffeomorphism covering the $\underline{\widetilde{\nabla}}$-exponential map $Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\colon T_{\underline{x}}(M) \to M$ at $\underline{x}$, where $T_x(X)$ is treated as a fibered manifold $Tp\colon T_x(X) \to T_{\underline{x}}(M)$.

If we compose $\big(Exp_x^{\widetilde{\nabla}}\big)^{-1}$ with a fibered linear isomorphism (fibered linear frame) $l\colon \mathbb{R}^{m,n} \to T_x(X)$ covering a linear frame $\underline{l}\colon \mathbb{R}^m \to T_{\underline{x}}(M)$, then we obtain a fibered $\widetilde{\nabla}$-normal coordinate system $\varphi = l^{-1} \circ \big(Exp_x^{\widetilde{\nabla}}\big)^{-1}\colon X \to \mathbb{R}^{m,n}$ with center x covering a $\underline{\widetilde{\nabla}}$-normal coordinate system

$$\underline{\varphi} = \underline{l}^{-1} \circ \big(Exp_{\underline{x}}^{\underline{\widetilde{\nabla}}}\big)^{-1}\colon M \to \mathbb{R}^m$$

with center $\underline{x}$. If $\varphi_1\colon X \to \mathbb{R}^{m,n}$ is another fibered $\widetilde{\nabla}$-normal coordinate system on X with center x, then $\varphi_1 = I \circ \varphi$ for some fibered linear isomorphism $I\colon \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$.

Quite similarly as above, if ∇ is a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) and $y \in Y$, then there exists a fibered-fibered ∇-normal coordinate system $\varphi\colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ with center y. If $\varphi_1\colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is another fibered-fibered ∇-normal coordinate system with center y, then $\varphi_1 = I \circ \varphi$ for some fibered-fibered linear isomorphism $I\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2}$.

4. The first main result. Let $Q^s_{\text{proj-proj}}$ be the vector space of all s-jets $j^s_{(0,0,0,0)}(\nabla)$ at $(0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ of projectable-projectable torsion-free classical linear connections ∇ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ satisfying the condition

$$\sum_{j,k=1}^{m_1+m_2+n_1+n_2} \nabla^i_{jk}(x)x^j x^k = 0 \quad \text{for} \quad i = 1, \dots, m_1+m_2+n_1+n_2,$$

where $\nabla^i_{jk} \colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}$, for $i,j,k = 1, \dots, m_1+m_2+n_1+n_2$, are the Christoffel symbols of a connection ∇ in the usual fibered-fibered coordinate system $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

Equivalently, $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ are fibered-fibered ∇-normal coordinates with center $(0,0,0,0)$. The equivalence is a simple consequence of the well-known equations of ∇-geodesics and the fact that in the ∇-normal coordinate system ∇-geodesics passing through the center are straight lines.

Let $\pi_s \colon Q^\infty_{\text{proj-proj}} \to Q^s_{\text{proj-proj}}$ for $s = 1, 2, \dots$ be the jet projections. Let $\pi^r_0 \colon J^r\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ be the jet projection. Denote $l_0 := j^1_{(0,0,0,0)}(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$.

Let $J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ be the fibre of π^r_0 at l_0. Let $\mu \colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$. We say that μ satisfies the finite determination property, if for any $\rho \in Q^\infty_{\text{proj-proj}}$ and $\sigma \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ we can find an open neighborhood $U \subset Q^\infty_{\text{proj-proj}}$ of ρ, open neighborhood $V \subset J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ of σ, a finite number s and a smooth map $f \colon \pi_s(U) \times V \to \mathbb{R}$ such that $\mu = f \circ (\pi_s \times id_V)$ on $U \times V$.

We are in a position to present the following example of the operator A in question.

Example 1. Let $\mu \colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ be a function satisfying the finite determination property. Given a projectable-projectable torsion-free classical linear connection ∇ on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2), we define the rth order Lagrangian

$$A^{\langle\mu\rangle}_Y(\nabla) \colon J^r\big(L^{\text{fib-fib}}(Y)\big) \to \wedge^m T^*(Y)$$

on $\pi_Y \colon L^{\text{fib-fib}}(Y) \to Y$ by

$$A^{\langle\mu\rangle}_Y(\nabla)(\sigma) := \mu\big(j^\infty_{(0,0,0,0)}(\varphi_*\nabla), J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma)\big) \cdot l^*_1 \wedge \dots \wedge l^*_{m_1+m_2+n_1+n_2},$$

where $m = m_1+m_2+n_1+n_2$, $\sigma \in J^r_l\big(L^{\text{fib-fib}}(Y)\big)$, $l = j^r_{(0,0,0,0)}(\varphi^{-1}) \in \big(L^{\text{fib-fib}}(Y)\big)_y$, $y \in Y$, $l_i = T(\varphi^{-1})\big(\frac{\partial}{\partial x^i}|_{(0,0,0,0)}\big)$ for $i = 1, \dots, m_1+m_2+n_1+n_2$ is the basis in $T_y(Y)$ and l^*_i for $i = 1, \dots, m_1+m_2+n_1+n_2$ is the dual basis in $T^*_y(Y)$ and $\varphi \colon Y \to \mathbb{R}^{m_1,m_2,n_1,n_2}$ is a fibered-fibered ∇-normal coordinate system with center y such that $J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma) \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$.

The definition of $A_Y^{\langle\mu\rangle}(\nabla)(\sigma)$ is correct because $germ_y(\varphi)$ is uniquely determined.

Consequently, for given a projectable-projectable torsion-free classical linear connection ∇ on Y, we have an rth order Lagrangian

$$A_Y^{\langle\mu\rangle}: J^r\big(L^{\text{fib-fib}}(Y)\big) \to \wedge^m T^*(Y), \quad \text{where } m = m_1 + m_2 + n_1 + n_2.$$

The family $A^{\langle\mu\rangle}: Q_\tau^{\text{proj-proj}} \rightsquigarrow \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big)$ of operators

$$\begin{aligned} A_Y^{\langle\mu\rangle}: Q_\tau^{\text{proj-proj}}(Y) &\longrightarrow Lagr^r\big(L^{\text{fib-fib}}(Y)\big), \\ \nabla &\longrightarrow A_Y^{\langle\mu\rangle}(\nabla) \end{aligned}$$

for any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-object Y is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator.

The main result of the present note is the following:

Theorem 1. *Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator*

$$A: Q_\tau^{\text{proj-proj}} \rightsquigarrow \big(J^r L^{\text{fib-fib}}, \wedge^m T^*\big),$$

where $m = m_1 + m_2 + n_1 + n_2$, is of the form $A = A^{\langle\mu\rangle}$ for a uniquely determined function $\mu: Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ satisfying the finite determination property.

Proof. Let A be a $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator in question. We must define a map $\mu: Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$ by

$$\mu\big(j^\infty_{(0,0,0,0)}(\nabla), \sigma\big) := \langle A_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(\nabla)(\sigma), (l_0)_1 \wedge \ldots \wedge (l_0)_{m_1+m_2+n_1+n_2}\rangle,$$

where $l_0 = ((l_0)_1, \ldots, (l_0)_{m_1+m_2+n_1+n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1+m_2+n_1+n_2})$. Then by the non-linear Peetre theorem [2], μ satisfies the finite determination property. By the invariance of A and $A^{\langle\mu\rangle}$ with respect to fibered-fibered normal coordinates we obtain $A = A^{\langle\mu\rangle}$. □

Remark 1. Quite similarly one can describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $A: Q_\tau^{\text{proj-proj}} \to \big(J^r L^{\text{fib-fib},k}, \wedge^m T^*\big)$ transforming projectable-projectable torsion-free classical linear connections ∇ on $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-objects Y into rth order Lagrangians $A_Y(\nabla)$ on $\pi^k_Y: L^{\text{fib-fib},k}(Y) \to Y$, where

$$\begin{aligned} &L^{\text{fib-fib},k}(Y) \\ &:= \Big\{ j^k_{(0,0,0,0)}(\psi) \mid \psi: \mathbb{R}^{m_1,m_2,n_1,n_2} \to Y \text{ is a local } \mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}\text{-map} \Big\} \end{aligned}$$

is the fibered-fibered kth order frame bundle for Y. All such natural operators in question are of the form $A^{\langle\mu\rangle}$ for functions

$$\mu: Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathbb{R}$$

satisfying the obviously modified finite determination property, where $J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)$ is the fiber of

$$J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$$

at the kth order frame $l_0 = j^k_{(0,0,0,0)}(id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}) \in L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})$. The generalized natural operators $A^{\langle\mu\rangle}$ can be defined quite similarly as in Example 1.

5. Euler morphisms and natural operators transforming connections into Euler morphisms. We recall that the rth order Euler morphism on a fibered manifold $p\colon X \to M$ is a base preserving morphism $E\colon J^r(X) \to V^*(X) \otimes \wedge^m T^*(M)$, where $m = \dim M$. Here $V^*(X)$ denotes the vector bundle dual to the vertical vector bundle $V(X)$ for X. Special Euler morphisms can be obtained from Lagrangians by means of the well-known Euler operator [2], [5].

Quite similarly as for Lagrangians, we can describe completely all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators

$$B\colon Q^{\text{proj-proj}}_{\tau} \to \big(J^r L^{\text{fib-fib}}, V^* L^{\text{fib-fib}} \otimes \wedge^m T^*\big),$$

where $m = m_1+m_2+n_1+n_2$, transforming projectable-projectable torsion-free classical linear connections ∇ on fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2) into rth order Euler morphisms $B_Y(\nabla)$ on

$$\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y.$$

6. The second main result.

Example 2. We consider a function

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \big(\mathcal{L}\big(G^1_{m_1,m_2,n_1,n_2}\big)\big)^*$$

satisfying the obviously modified finite determination property, where $\mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$ denotes the Lie algebra of Lie group $G^1_{m_1,m_2,n_1,n_2}$. Given a projectable-projectable torsion-free classical linear connection on a fibered-fibered manifold Y of dimension (m_1, m_2, n_1, n_2), we define an rth order Euler morphism $B^{\langle\mu\rangle}_Y(\nabla)\colon J^r\big(L^{\text{fib-fib}}(Y)\big) \to V^*\big(L^{\text{fib-fib}}(Y)\big) \otimes \wedge^m T^*(Y)$, where $m = m_1 + m_2 + n_1 + n_2$, on $\pi_Y\colon L^{\text{fib-fib}}(Y) \to Y$ by

$$\begin{aligned}&\big\langle B^{\langle\mu\rangle}_Y(\nabla)(\sigma), \eta^*_{|l}\big\rangle\\ &\quad = \Big\langle \mu(j^\infty_{(0,0,0,0)}(\varphi_*\nabla), J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma)), \eta\Big\rangle\, l^*_1 \wedge \ldots \wedge l^*_{m_1+m_2+n_1+n_2}\end{aligned}$$

for all $\sigma \in \big(J^r_l\big(L^{\text{fib-fib}}(Y)\big)$, $l = (l_1, \ldots, l_m) \in \big(L^{\text{fib-fib}}(Y)\big)_y$, $y \in Y$, where $m = m_1 + m_2 + n_1 + n_2$, $\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$, where η^* is the (vertical) fundamental vector field on the principal $G^1_{m_1,m_2,n_1,n_2}$-bundle $L^{\text{fib-fib}}(Y)$ corresponding to η and $l^*_1, \ldots, l^*_m \in T^*_y Y$ is the dual basis to $l_1, \ldots, l_m \in T_y Y$

and $\varphi\colon Y \to \mathbb{R}^{m_1 m_2, n_1, n_2}$ is a fibered-fibered ∇-normal coordinate system on Y with center y such that $\varphi(y) = (0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$ and $J^r\big(L^{\text{fib-fib}}(\varphi)\big)(\sigma) \in \big(J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big)\big)$. The correspondence $B^{\langle\mu\rangle}\colon Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, V^*L^{\text{fib-fib}} \otimes \wedge^m T^*\big)$, where $m = m_1 + m_2 + n_1 + n_2$, is $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator.

Similarly as Theorem 1 one can prove the following:

Theorem 2. *Any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator*

$$B\colon Q^{\text{proj-proj}}_\tau \rightsquigarrow \big(J^r L^{\text{fib-fib}}, V^*L^{\text{fib-fib}} \otimes \wedge^m T^*\big),$$

where $m = m_1 + m_2 + n_1 + n_2$, is of the form $B = B^{\langle\mu\rangle}$ for some uniquely determined function

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to (\mathcal{L}(G^1_{m_1,m_2,n_1,n_2}))^*$$

satisfying the modified finite determination property.

Proof. Similarly as in the proof of Theorem 1 we define

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to (\mathcal{L}(G^1_{m_1,m_2,n_1,n_2}))^*$$

by

$$\begin{aligned}&\langle \mu(j^\infty_{(0,0,0,0)}(\nabla), \sigma), \eta\rangle \\ &\quad = \langle B_{\mathbb{R}^{m_1,m_2,n_1,n_2}}(\nabla)(\sigma), \eta^*_{l_0} \otimes (l_0)_1 \wedge \ldots \wedge (l_0)_{m_1+m_2+n_1+n_2}\rangle,\end{aligned}$$

where

$$\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2}),\ j^\infty_{(0,0,0,0)}(\nabla) \in Q^\infty_{\text{proj-proj}},\ \sigma \in J^r_{l_0}\big(L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big),$$

η^* is the fundamental vector field on $L^{\text{fib-fib}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ corresponding to $\eta \in \mathcal{L}(G^1_{m_1,m_2,n_1,n_2})$ and $l_0 = ((l_0)_1, \ldots, (l_0)_{m_1+m_2+n_1+n_2})$ is the basis in $T_{(0,0,0,0)}(\mathbb{R}^{m_1+m_2+n_1+n_2})$. Then $B = B^{\langle\mu\rangle}$. □

Remark 2. Quite similarly one can describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $B\colon Q^{\text{proj-proj}}_\tau \to (J^r L^{\text{fib-fib},k}, V^*L^{\text{fib-fib},k} \otimes \wedge^m T^*)$, where $m = m_1+m_2+n_1+n_2$, transforming projectable-projectable torsion-free classical linear connections ∇ on (m_1, m_2, n_1, n_2)-dimensional fibered-fibered manifolds Y into Euler morphisms $B_Y(\nabla)$ on $\pi^k_Y\colon L^{\text{fib-fib},k}(Y) \to Y$ of fibered-fibered frames of order k of Y. All such natural operators are of the form $B^{\langle\mu\rangle}$ for all

$$\mu\colon Q^\infty_{\text{proj-proj}} \times J^r_{l_0}\big(L^{\text{fib-fib},k}(\mathbb{R}^{m_1,m_2,n_1,n_2})\big) \to \mathcal{L}(G^k_{m_1,m_2,n_1,n_2})$$

satisfying the obviously modified finite determination property. The natural operators $B^{\langle\mu\rangle}$ can be constructed similarly as in Example 2.

References

[1] Kurek, J., Mikulski, W. M., *Lagrangians and Euler morphisms from connections on the frame bundle*, Proceedings of the XIX International Fall Workshop on Geometry and Physics, Porto, 2010.
[2] Kolář, I., Michor, P. W. and Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.
[3] Kolář, I., *Connections on fibered squares*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **59** (2005), 67–76.
[4] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry, Vol. I*, Interscience Publisher, New York–London, 1963.
[5] Kurek, J., Mikulski, W. M., *On the formal Euler operator from the variational calculus in fibered-fibered manifolds*, Proc. of the 6 International Conference Aplimat 2007, Bratislava, 223–229.

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